Homework for Real Analysis(V)

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1 Answer

1.

证明. 注意到由绝对连续性, 我们有

$$f(x) - f(a) = \int_a^x f'(t)dt. \tag{1.1}$$

又 $f(a) = 0$, 故

$$f(x) = \int_a^x f'(t)dt. \tag{1.2}$$

代入即得

$$\begin{aligned}\int_a^b |f'(x)f(x)|dx &= \int_a^b |f'(x)\int_a^x f'(t)dt|dx \\ &\leq \int_a^b \int_a^b \chi_{\{t\geq x\}}|f'(t)f'(x)|dtdx\end{aligned} \tag{1.3}$$

注意到由 Fubini 定理与对称性, 上式将 t 和 x 交换后不变, 故

$$\begin{aligned}\int_a^b \int_a^b \chi_{\{t\geq x\}}|f'(t)f'(x)|dtdx &= \frac{1}{2}\left[\int_a^b \int_a^b \left(\chi_{\{t\geq x\}} + \chi_{\{t\leq x\}}\right)|f'(t)f'(x)|dtdx\right] \\ &= \frac{1}{2}\left[\int_a^b \int_a^b |f'(t)f'(x)|dtdx\right] \\ &= \frac{1}{2}\left(\int_a^b |f'(x)|dx\right)^2,\end{aligned} \tag{1.4}$$

即证毕. □

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2. 第二题除了几乎处处收敛外, 满足一些其他的条件也可保证弱收敛:

Theorem 1.1. *Let* (f_n) *be a sequence in* $L^p, 1 < p \leq \infty$ *such that some of the following assumptions hold,*

(i) $\sup_n \|f_n\|_p \leq M$.

(ii) $f_n \to f$ *a.e.* $x \in \Omega$.

(ii') $\|f_n - f\|_1 \to 0, n \to \infty$.

(ii') (f_n) *converges in measure to* f.

If (i) holds and one of (ii)-(ii') holds, then $f_n \rightharpoonup f$ *weakly* $\sigma\left(L^p, L^{p'}\right)$.

注 1.1. 一些同学直接使用控制收敛定理, 这是不对的, 注意到条件 *(i)* 不能保证 $\sup_n |f_n| \in L^p$, 这个要成立还需要满足一些极大函数的不等式 *(*比如有点像鞅的 *Doob* L^p *maximal inequality* 的东西*)*.

满足条件 (i) 和 (ii) 的证明是关键的, 其余的由依测度收敛的性质取子列即得. 这个有两种证明, 第一种证明是简单的, 首先考虑有限测度空间, 之后再用逼近的思想 (由于 $\mathbb{R}^d$ 是 σ 有限的测度空间).

证明. 由 f_n 的一致有界性, 以及 $f_n \to f$ a.e. 由 Fatou 引理有:

$$\int_\Omega |f|^p d\mu \leq \liminf_{n\to\infty} |f_n|^p d\mu \leq M, \tag{1.5}$$

故有 $f \in L^p$. 要证弱收敛即证对任意的 $g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$ 都有

$$\int_\Omega (f_n - f) g d\mu \to 0, (n \to \infty). \tag{1.6}$$

一方面由于 $f, f_n \in L^p, g \in L^q$ 且测度有限, 由 Hölder 不等式知 $f, f_n, g \in L^1$, 故

$$\int_\Omega |g| d\mu < M_1. \tag{1.7}$$

一方面由 $f_n \to f$ a.e. 由 Egorv 定理知 $\forall \delta > 0, \exists E_\delta \subset \Omega$, 使得 $\mu(E_\delta) < \delta$ 且在 $\Omega \setminus E_\delta$ 上 $f_n \Longrightarrow f$. 另一方面由于 $g \in L^q$ 故由积分的绝对连续性, $\forall \epsilon > 0, \exists \delta > 0$ 使得只要 $\mu(E_\delta) < \delta$

$$\int_{E_\delta} |g|^q d\mu < \epsilon. \tag{1.8}$$

故取 Egorv 定理的 E_δ 即可. 故有

$$\begin{aligned}\int_\Omega (f_n-f)g d\mu &\le \int_\Omega |(f_n-f)||g|d\mu \\ &\le \int_{E_\delta} |(f_n-f)||g|d\mu + \int_{\Omega\setminus E_\delta} |(f_n-f)||g|d\mu \\ &\le \left(\int_{E_\delta} |(f_n-f)|^p d\mu\right)^{\frac{1}{p}} \left(\int_{E_\delta} |g|^q d\mu\right)^{\frac{1}{q}} + \int_{\Omega\setminus E_\delta} |(f_n-f)||g|d\mu. \end{aligned} \tag{1.9}$$

由一致收敛性故

$$\int_{\Omega\setminus E_\delta} |(f_n-f)||g|d\mu < \epsilon M_1. \tag{1.10}$$

再由积分的绝对收敛性则有

$$\left(\int_{E_\delta} |(f_n-f)|^p d\mu\right)^{\frac{1}{p}} \left(\int_{E_\delta} |g|^q d\mu\right)^{\frac{1}{q}} < 2M\epsilon^{\frac{1}{q}}. \tag{1.11}$$

故令 $\epsilon \to 0$ 即得. 对于测度无穷的空间, 请考虑 $\Omega \cap B(0,k)$ 和 $\Omega \cap B(0,k)^c$, 则由刚刚的证明知道

$$\int_\Omega (f_n-f)g\chi_{B(0,k)}d\mu \to 0, (n\to\infty). \tag{1.12}$$

对于后者 $\Omega \cap B(0,k)^c$, 和刚刚的估计一样, 注意到 $g \in L^q$, 且 $\mu(\Omega\cap B(0,k)^c) \to 0$ 当 $k\to\infty$ 时, 故

$$\int_\Omega (f_n-f)g\chi_{B(0,k)^c}d\mu \to 0, (n\to\infty). \tag{1.13}$$

证毕. □

第二种证明使用两个引理 (周民强, P284, 285 定理 6.26 和定理 6.28), 由定理 6.28 知若满足条件 (i) 和 (ii) 的, 则其一定存在一个子列 f_{nk} 在 L^p 中弱收敛, 又 $f_n \to f$ a.e., 由定理 6.26 知极限唯一, 故 f_{nk} 弱收敛到 f. 即对 f_n 的任一子列, 都存在一个子列的子列弱收敛, 则立得 f_n 弱收敛 (定理 6.26 中的有限测度同样并不需要, 由于 $\mathbb{R}^d$ 是 σ 有限的测度空间).

注 1.2. $p=1$ 时定理不对, 考虑 $\Omega = \mathbb{R}^+$ 和

$$f_n(x) = \frac{1}{n}\chi_{(0,n)}(x), g(x) = 1. \tag{1.14}$$

则我们有 $\|f_n\| = 1$ 且 $f_n \to 0$ *a.e.* 但是

$$\int_{R^+} f_n(x)g(x)dx = \int_0^n \frac{1}{n}dx = 1 \nrightarrow 0. \tag{1.15}$$

3. 将 $\mathbb{Q}$ 排序为 $\{q_n\}$, 则定义如下函数即满足条件

$$f(x) := \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{[q_k,\infty)}(x). \tag{1.16}$$

请大家自己验证.

4. $T_F([a,b]) \le \int_a^b |F'(x)|dx$ 是显然的. 利用 $T_F(a,x) = P_F(a,x) + N_F(a,x)$ 另一边不等号即得.

5. 之前给大家总结过, 对于一个度量空间 $(\mathcal{X}, d)$ 来说, 如果存在不可数子集 $A \subset \mathcal{X}$, $\exists d_0 > 0$ 使得对 $\forall x, y \in A (x \neq y)$, 有 $d(x,y) \ge d_0$, 则度量空间不可分. 故考虑 $f_t(x) = \chi_{(0,t)}(x), 0 < t < \infty$ 即可, 这里的度量空间即 $(L^\infty, \|\cdot\|_\infty)$.

注 1.3. *如何证明 l^∞ 是不可分的?*

6. 按定义验证即可.

7. (a) 设 f_n 是柯西列, 则 f_n 在 $C([0,1])$ 是柯西列且 f_n' 在 L^1 空间是柯西列. 故知存在 $f(x)$ 使得 $f_n(x)$ 一致收敛到 $f(x)$, 且存在 $v(x)$ 使得 $f_n'(x)$ 在 L^1 收敛到 $v(x)$. 又我们知 $f_n(x) - f_n(0) = \int_0^x f_n'(t)dt$, 故取极限有 $f(x) - f(0) = \int_0^x v(t)dt$, 故有 $f' = v \in L^1$. 故 f 绝对连续, 且 f_n 依 AC 范数收敛到 f.

(b) 由 $L^1([0,1])$ 可分, 故有可数稠密子集 ϕ_k, 则 AC 中的可数稠密子集定义为 $\psi_k := \int_0^x \phi_k$. 大家可以验证这是一个 AC 中的可数稠密子集.

8. 利用可测集 $E \subset \mathbb{R}$ 可以分解为可数紧集的并和一个零测集的并, 由于绝对连续, 故零测集的像集还是零测, 连续函数把紧集映射为紧集, 故可测.

9. 几个关键的证明点在于首先证明 F 绝对连续, 再由可积性可以推出 $\lim\limits_{x\to\infty} F(x) = 0$, 即得.

10.

$$g := f\chi_{\|f\|\ge 1}, h := f\chi_{\|f\|<1}. \tag{1.17}$$

11. 利用 Hölder 不等式和 Tonelli 定理即得, 证明见周民强 P278 定理 6.21.

12. 该问等价于证明如下定理:

Theorem 1.2. *设 $\{f_n\}$ 为 Hilbert 空间中的序列, 则 $\|f_n - f\|_2 \to 0$ 当且仅当*

1. *$\{f_n\}$ 弱收敛到 f.*
2. *$\|f_n\|_2 \to \|f\|_2$.*

注 1.4. *Hilbert 空间有很多很好的性质, 比如其具有内积结构, 而且他是一个自对偶空间 (Risez 表示定理), 所以这个定理实际上很好证. 但这个定理在 $L^p(1 < p < \infty)$ 也成立, 此即 Radon 定理 (周民强 P286, 定理 6.29), 这个时候证明不是很容易了.*

证明. ($\Longrightarrow$) 这个是容易的, 第一个是由于强收敛故一定有弱收敛成立:

$$|g(f_n) - g(f)| \leq \|g\|\|f_n - f\|, \tag{1.18}$$

这里是用了 Cauchy-Schwarz 定理, 对于一般的 Banach 空间来说 $\|g\|$ 表示的是线性算子的范数, 第二个由三角不等式立得.

($\Longleftarrow$)

$$\begin{aligned}\|f_n - f\|_2^2 &= \langle f_n - f, f_n - f\rangle \\ &= \|f_n\|_2^2 - 2\langle f_n, f\rangle + \|f\|_2^2.\end{aligned} \tag{1.19}$$

由弱收敛知上式内积那一项趋于 $\|f\|_2^2$, 再有范数收敛, 故上式收敛于 0. □

注 1.5. *关于无穷维空间如 L^p 空间或者是 Hilbert 空间中这些收敛的关系, 大家在泛函分析会更加深入的学习, 这部分可参考 Brezis 的 Functional Analysis, Sobolev Spaces and Partial Differential Equations, 或者是张恭庆老师写的泛函分析 (上册)/(下册). 事实上更一般的, 我们可以在一个拓扑向量空间 (Topological Vector Space, TVS) 考虑这些关系, 因为本质上来说我们只需要拓扑和线性运算相容就可以了, 这部分内容大家可以参考 Rudin 的 Functional Analysis, 或者是 Folland 的 Real Analysis. 现在实分析我们学习了在有限维空间构造 Lebsegue 测度, 类似的我们可以考虑那无穷维空间上怎么构造测度呢? 无穷维空间上测度的构造和随机过程的构造是非常有关系的, 比如 $C([0,T])$ 上 Wiener 测度的构造就对应着 $\mathbb{R}$ 上的布朗运动, Hilbert 空间上测度的构造这对应的是无穷维空间的布朗运动.*

最后附录里面总结了在 L^p 空间上强收敛, 弱收敛, 依范数收敛的一些关系, 这个文档是由我本科的实分析的助教也是我的学长, 毕铖整理的, 供大家参考.

附录

L^p空间中函数列收敛型的总结

这篇文章主要总结了一下Lp空间中函数列几乎处处收敛、强收敛、弱收敛、范数收敛、一致收敛、依测度收敛等收敛方式的关系以及它们的证明。

其中有几个定理我们给出了多种不同的证明方法，在一些p的范围的端点，比如1和∞，如果定理不成立我们给出了反例（由于一开始写的时候就是用英语写的，现在也就不再花时间翻译回来了，请忽略我简单粗暴的英文表达!!!!!!），大家有兴趣可以看一下里面的各种结论，其中有一些结论是取自 H.Brezis 的泛函分析书。

1 Notations

Let Ω be a σ-finite measure space. Denote by $L^p(\Omega)$ the Lebesgue spaces equipped with the norm $||\cdot||_p$ and the Lebesgue measure μ, $1 \le p \le \infty$.

Definition 1.1 (Weak convergence in L^p, [3]) *Let* $1 \le p, q \le \infty$, $1/p + 1/q = 1$, $f \in L^p(\Omega)$, $f_n \in L^p(\Omega)$ $(n \in \mathbb{N})$, *if*

$$\lim_{n\to\infty} \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx, \ \forall\, g \in L^q(\Omega), \tag{1.1}$$

then we say that (f_n) *converges weakly to* f.

For simplicity, we shall use the following notations describing the convergence.

(1) " $f_n \to f \ a.e. \ x \in \Omega$ " : almost everywhere convergence on Ω.

(2) " $f_n \rightharpoonup f$ weakly $\sigma(L^p, L^{p'})$ " : weak convergence in L^p.

(3) " $f_n \to f$ strongly in L^p " : $||f_n - f||_p \to 0$ as $n \to \infty$.

2 Summerization of relationships

Lemma 2.1 *Suppose that* (f_n) *is a sequence in* $L^p(\Omega)$, *and the following conditions hold,*

(a) $f_n \rightharpoonup f$ *weakly* $\sigma(L^p, L^{p'})$,

(b) $f_n \to g \ a.e. \ x \in \Omega$,

then $f = g, \ a.e.$ *in* Ω.

Proof 1 of Lemma 2.1. In view of [3, Theorem 6.26] and Ω is a σ-finite measure space, the proof is completed. □

Proof 2 of Lemma 2.1. Define $K_n = \overline{\text{conv}\Big(\bigcup_{i=n}^{\infty}\{f_i\}\Big)}$. By [1, Exercise 3.13] we know $f \in K_n$. Thus, there exists $g_n \in \text{conv}\Big(\bigcup_{i=n}^{\infty}\{f_i\}\Big)$ such that $||g_n - f||_p < \dfrac{1}{n}$ and therefore we shall choose $\{g_{n_k}\}$ such that $g_{n_k} \to f\ a.e.$ in Ω. In view of (b), for almost everywhere $x \in \Omega$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$, $|f_n(x) - g(x)| < \varepsilon$ whenever $n \geq N$. Furthermore, there exist finite sets $I_n \subseteq \{n, n+1, ...\}$ and $\{\lambda_i\}_{i\in I_n} \subset [0,1]$ such that

$$g_n(x) = \sum_{i\in I_n} \lambda_i f_i(x),\ \sum_{i\in I_n} \lambda_i = 1.$$

Through an easy computation we obtain that

$$\Big|g_n(x) - g(x)\Big| = \Big|\sum_{i\in I_n} \lambda_i f_i(x) - \sum_{i\in I_n} \lambda_i g\ (x)\Big| = \Big|\sum_{i\in I_n} \lambda_i (f_i(x) - g(x))\Big| \leq \varepsilon \sum_{i\in I_n} \lambda_i = \varepsilon,$$

which implies that $g_n \to g\ a.e.\ x \in \Omega$, so $f = g\ a.e. x \in \Omega$. And the proof is completed. □

Theorem 2.2 *Let (f_n) be a sequence in L^p, $1 < p \leq \infty$ such that some of the following assumptions hold,*

(i) $||f_n||_p \leq M$.

(ii) $f_n \to f\ a.e.\ x \in \Omega$.

(ii′) $||f_n - f||_1 \to 0$, $n \to \infty$.

(ii″) (f_n) *converges in measure to* f.

If (i) holds and one of (ii)-(ii″) holds, then $f_n \rightharpoonup f$ weakly $\sigma(L^p, L^{p'})$.

Proof. We mainly prove the theorem under condition (i) and (ii). Since if (ii′) or (ii″) holds, for any subsequence of (f_n), there exist its subsequence almost everywhere converges to f.

Case 1 $1 < p < \infty$. By (i), (ii) and Lemma 2.1, consider an arbitrary subsequence (f_{n_k}) of (f_n), going if necessary to a subsequence, we can assume that $f_{n_k} \rightharpoonup f$ weakly $\sigma(L^p, L^{p'})$. Therefore, $(f_n) \rightharpoonup f$ weakly $\sigma(L^p, L^{p'})$. For an alternative method we refer to [5, Theorem 2.3.17].

Case 2 $p = \infty$. Given $g \in L^1(\Omega)$, it is easy to know that $||f||_\infty \leq M$ and $f_n g \to fg\ a.e.\ x \in \Omega$. Noting that $|(f_n - f)g| \leq 2M|g| \in L^1(\Omega)$, then by the dominated convergence theorem we know the proof is completed.

□

Remark 2.3 *Theorem 2.2 fail if $p = 1$, that is, in this time condtions (i) and (ii) do not imply $f_n \rightharpoonup f$ weakly $\sigma(L^p, L^{p'})$. We may construct the counterexample as follows, taking $\Omega = \mathbb{R}^+$ and*

$$f_n(x) = \frac{1}{n}\boldsymbol{\chi}_{(0,n)}(x),\ g(x) = 1.$$

then we have $||f_n||_1 = 1$ and $f_n \to f = 0\, a.e.\, x \in \Omega$, however, $\forall n \in \mathbb{N}_+$,

$$\int_{\mathbb{R}^+} f_n(x)g(x)dx = \int_0^n \frac{1}{n}dx = 1 \nrightarrow 0.$$

Theorem 2.4 *Let (f_n) be a sequence in L^p, $1 \leq p \leq \infty$ such that some of the following assumptions hold,*

(i) $f_n \to f\ a.e.\, x \in \Omega$.

(ii) *(f_n) converges in measure to f.*

(iii) $p \in (1, \infty]$, $||f_n||_p \leq M$ *and* $\mu(\Omega) < \infty$.

(iv) $p \in [1, \infty)$, $||f_n||_p \to ||f||_p < \infty$.

(v) $p \in [1, \infty)$, *and there exists $F \in L^p$ such that* $|f_n(x)| \leq F(x)$, $a.e. x \in \Omega$.

If (i),(iii) or (ii),(iii) hold, then $f_n \to f$ strongly in L^q, $1 \leq q < p$. If (i),(iv) or (i),(v) hold, then $f_n \to f$ strongly in L^p.

Proof.

Step 1. We first assume (i),(iii) hold, if $1 < p < \infty$, given $\varepsilon > 0$, Egorov's theorem guarantees the existence of a measurable subset $B \subset \Omega$, such that $\mu(B) < \varepsilon$ and $f_n \to f$ uniformly on $\Omega \backslash B$. Therefore, there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have

$$\left(\int_{\Omega \backslash B} \left| f_n - f \right|^q dx \right)^{\frac{1}{q}} < \varepsilon,\ \forall q \in [1, p).$$

By applying Hölder inequality we obtain that

$$\begin{aligned} \int_B \left| f_n - f \right|^q dx &\leq \mu(B)^{\frac{p-q}{p}} \left(\int_B \left| f_n - f \right|^p dx \right)^{\frac{q}{p}} \\ &\leq \mu(B)^{\frac{p-q}{p}} \left| \left| f_n - f \right| \right|_p^q \\ &\leq \mu(B)^{\frac{p-q}{p}} \Big(||f||_p + M \Big)^q. \end{aligned}$$

Then we infer from above that $||f_n - f||_q \to 0$, as $n \to \infty$.

If $p = \infty$, we only need to note that $|f_n - f|^q \leq (2M)^q \in L^1(\Omega)$ and the conclusion can be derived by the dominated convergence theorem.

Step 2. Then if (ii),(iii) hold, then there are various ways to testify the problem.

Approach 1 We only consider $p = \infty$, and it is analogous in other cases. Given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$,

$$\mu(E_n) := \mu\Big(\big\{ x \in \Omega : |f_n(x) - f(x)| > \varepsilon \big\} \Big) < \varepsilon$$

whenever $n \geq N$, we obtain that

$$\int_{\Omega} |f_n - f|^r dx = \int_{\Omega \setminus E_n} |f_n - f|^r dx + \int_{E_n} |f_n - f|^r dx$$
$$\leq \varepsilon^r \cdot \mu(\Omega) + (2M)^r \cdot \varepsilon.$$

Then the proof is completed.

Approach 2 By (ii), for any subsequence of (f_n), there exist its subsequence almost everywhere converges to f. Then use the conclusion in **Step 1**.

Step 3. Next, we prove Theorem 2.4 under assumptions (i) and (iv). We present several different methods for it.

Approach 1 By Brezis-Lieb's lemma [2, Lemma 1.32] we immediately obtain the conclusion.

Approach 2 It is obvious that $\{||f_n||_p\}_{n\geq 1}$ are bounded, by Theorem 2.1 and Radon's theorem, the proof is completed. (This method hold true only for $p \in (1, \infty)$).

Approach 3 By Fatou's lemma we observe that

$$\big|f_n - f\big|^p \leq 2^{p-1}\Big(|f_n|^p + |f|^p\Big) =: F_n(x),$$

therefore we know that $f \in L^p$. Futhermore, we can easily obtain that

(i) $F_n(x) \to F(x) := 2^p|f|^p,\ a.e.\, x \in \Omega.$

(ii) $\lim\limits_{n\to\infty} \int_{\Omega} F_n(x)dx = \int_{\Omega} F(x)dx.$

Thus, by the general dominated convergence theorem, $f_n \to f$ strongly in L^p. Then the proof is completed.

Approach 4 We first claim that $\forall\, E \subset \Omega$, the limit

$$\lim_{n\to\infty} \int_E |f_n|^p dx = \int_E |f|^p dx$$

holds. In fact, since

$$\begin{aligned}
\varliminf_{n\to\infty} \int_E |f_n|^p dx \geq \int_E |f|^p dx &= \int_{\Omega} |f|^p dx - \int_{E^c} |f|^p dx \\
&\geq \int_{\Omega} |f|^p dx - \varliminf_{n\to\infty} \int_{E^c} |f_n|^p dx \\
&= \lim_{n\to\infty} \int_{\Omega} |f|^p dx + \varlimsup_{n\to\infty} \int_{E^c} \Big(-|f_n|^p\Big) dx \\
&= \varlimsup_{n\to\infty} \int_E |f_n|^p dx.
\end{aligned}$$

Then given $\varepsilon > 0$, there exists $A \subset \Omega$, $\mu(A) < \infty$ and

$$\int_{\Omega\cap A^c} |f|^p dx < \frac{\varepsilon}{5\cdot 2^p}.$$

For the fixed $\varepsilon > 0$, $\exists\,\delta > 0$, $\forall E \subset \Omega$ and $\mu(E) < \delta$, $\int_E |f|^p dx < \frac{\varepsilon}{5}$, by Egorov's theorem, $\exists\, B \subset A$ such that $\mu(A \cap B^c) < \delta$ and $f_n \to f$ uniformly on B. Therefore, $\exists\, N \in \mathbb{N}_+$, if $n \geq N$,

$$\int_B |f_n - f|^p dx < \frac{\varepsilon}{5} \quad \text{and} \quad \int_{A\cap B^c} |f|^p dx < \frac{\varepsilon}{5\cdot 2^p},$$

then we have that

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_\Omega \left|f_n - f\right|^p dx &= \overline{\lim_{n\to\infty}} \left(\int_{\Omega\cap A^c} + \int_B + \int_{A\cap B^c} \right) \\ &\leq 2^p \int_{\Omega\cap A^c} \Big(|f_n|^p + |f|^p\Big) dx + 2^p \int_{A\cap B^c} \Big(|f_n|^p + |f|^p\Big) dx + \frac{\varepsilon}{5} \\ &\leq 2^p \cdot \frac{\varepsilon}{5\cdot 2^p} \cdot 4 + \frac{\varepsilon}{5} = \varepsilon.\end{aligned}$$

The proof is ended since $\varepsilon > 0$ is arbitrary.

Approach 5 We refer to [4, Page 306].

□

Remark 2.5 **(1)** *The conclusion of (i) and (iii) in Theorem 2.4 may fail if $q = p$, that is, in this time (i),(iii) do not infer $f_n \to f$ strongly in L^p. Assume that $\Omega = (0,1)$ and*

$$f_n(x) = \begin{cases} \sqrt{n}, & x \in \left(0, \frac{1}{n}\right), \\ 0, & x \in \left[\frac{1}{n}, 1\right). \end{cases}$$

then $f_n \to 0,\ a.e. x \in (0,1)$, however, $||f_n - 0||_2 = 1 \nrightarrow 0$. If $q = p = \infty$, let $\Omega = (0,1)$ and $f_n(x) = x^n$, $n \geq 1$, then

$$f_n \to 0,\ a.e. x \in (0,1), \quad \textit{but} \quad ||f_n - 0||_\infty = 1,\ \forall\, n \in \mathbb{N}_+.$$

(2) *The condition $\mu(\Omega) < \infty$ in (iii) cannot be dropped, or we consider $\Omega = \mathbb{R}^+$, $p = \infty$, $q = 1$ and $f_n(x) = \chi_{(0,n)}(x)$, then $f_n \to f = 1,\ a.e. x \in \Omega$ but $||f_n - f||_1 = \infty$, $\forall\, n \in \mathbb{N}_+$.*

(3) *The condition $f \in L^p$ in (iv) cannot be dropped. Otherwise, considering that $\Omega = \mathbb{R}^+$ and $f_n(x) = \chi_{(0,n)}(x)$, $f(x) = 1$, then $||f_n||_p \to ||f||_p = \infty$, but*

$$||f_n - f||_p^p = \int_{\mathbb{R}^+} \chi^p_{(n,\infty)}(x) dx = \infty,\ \forall\, n \in \mathbb{N}_+.$$

(4) *The conclusion of Theorem 2.4 may fail if $p = \infty$ in (iv) or (v). We shall consider a similar counterexample as in (2) and take $F(x) = 1$.*

(5) *The conclusion of (ii) and (iii) in Theorem 2.4 may fail if $\mu(\Omega)=\infty$, we consider $1<p<\infty$, $\Omega=\mathbb{R}^+$ and*

$$f_n(x)=n^{-\frac{1}{p}}\chi_{(0,n)}(x),$$

then (f_n) converges in measure to $f=0$ and $||f_n||=1$, but $||f_n-f||_r^r=n^{1-\frac{r}{p}}\to\infty$.

Theorem 2.6 *Let (f_n) be a sequence in $L^p\cap L^q$, $f\in L^p$, $1\le p,q\le\infty$. Futhermore,*

$$f_n\to f \ \text{ strongly in } \ L^p,\ ||f_n||_q\le M, \tag{2.1}$$

then $f\in L^r$ and $f_n\to f$ strongly in L^r, for every r between p, $r\neq q$.

Proof. By inequality

$$\big|\big|f_n\big|\big|_r\le\big|\big|f_n\big|\big|_p^{\theta}\big|\big|f_n\big|\big|_q^{1-\theta},\ \ \frac{1}{r}=\frac{\theta}{p}+\frac{1-\theta}{q},\ \theta\in[0,1]$$

we immediately know that $\sup\limits_{n\ge1}\left\{\,||f_n||_r\,\right\}<\infty$. From Fatou's lemma we can see that $f\in L^r$ for r between p and q. If $\theta\neq0$, then

$$\begin{aligned}\big|\big|f_n-f\big|\big|_r&\le\big|\big|f_n-f\big|\big|_p^{\theta}\big|\big|f_n-f\big|\big|_q^{1-\theta}\\&\le\big|\big|f_n-f\big|\big|_p^{\theta}\cdot\Big(\big|\big|f_n\big|\big|_q+\big|\big|f\big|\big|_q\Big)^{1-\theta}\to0,\ \text{as}\ n\to\infty,\end{aligned}$$

which ends the proof. □

Theorem 2.7 (Radon) *Let (f_n) be a sequence in L^p, $1<p<\infty$, if $f_n\rightharpoonup f$ weakly $\sigma(L^p,L^{p'})$ and $||f_n||_p\to||f||_p$, $n\to\infty$. Then $f_n\to f$ strongly in L^p.*

Proof. For an alternative approach one can see [3, Theorem 6.29]. Here we present another method. From the following Clarkson's first and second inequality,

$$\left|\left|\frac{f+g}{2}\right|\right|_p^p+\left|\left|\frac{f-g}{2}\right|\right|_p^p\le\frac{1}{2}\Big(||f||_p^p+||g||_p^p\Big),\ \forall\, f,g\in L^p,\ 2\le p<\infty.$$

$$\left|\left|\frac{f+g}{2}\right|\right|_p^{p'}+\left|\left|\frac{f-g}{2}\right|\right|_p^{p'}\le\Big(\frac{1}{2}||f||_p^p+\frac{1}{2}||g||_p^p\Big)^{\frac{1}{p-1}},\forall\, f,g\in L^p,\ 1<p\le2.$$

We infer that L^p spaces are uniformly convex for $1<p<\infty$. Then by [1, Proposition 3.32] we know that the proof is ended. □

Remark 2.8 *Radon's theorem may fail if $p=1,\infty$. For counterexamples, see [3, Page 287].*

Theorem 2.9 *Assume that $\mu(\Omega)<\infty$, for $1\le p\le\infty$, if (f_n) converges to f uniformly in L^p, then $f_n\to f$ strongly in L^p.*

Proof. We only need to note that

$$\int_\Omega\big|f_n-f\big|^p dx\le\sup_{x\in\Omega}|f_n(x)-f(x)|^p\cdot\mu(\Omega)\to0$$

as $n\to\infty$ if $1\le p<\infty$. □

Theorem 2.10 *Suppose that (f_n) are measurable functions on Ω such that*

$$f_n(x) \geq f_{n+1}(x) \, \Big(f_n(x) \leq f_{n+1}(x)\Big), \, k \in \mathbb{N}_+$$

hold, then (f_n) converges in measure to a measurable function f if and only if $f_n(x) \to f(x)$, a.e.$x \in \Omega$.

Proof. The conclusion can be obtained by Riesz's theorem and the monotonicity of (f_n). □

Theorem 2.11 *Let (f_n) be a sequence in $L^1(\Omega)$ and $f \in L^1(\Omega)$, for all measurable subset $E \subset \Omega$, if*

$$\int_E f_n(x)dx \leq \int_E f_{n+1}(x)dx, \, \forall \, n \in \mathbb{N}_+,$$

then

$$\lim_{n\to\infty} f_n(x) = f(x), \, a.e.x \in \Omega$$

if and only if

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Proof. See [3, Exercise 4.5, 4.23]. □

Theorem 2.12 *Let (f_n) be a sequence in $L^p(\Omega)$ and $f \in L^p(\Omega)$, then*

(i) *if $1 < p \leq \infty$, then the following properties are equivalent,*

(A) $f_n \rightharpoonup f$ *weakly* $\sigma(L^p, L^{p'})$.

(B) $||f_n||_p \leq M$ *and* $\int_E f_n(x)\,dx \to \int_E f(x)\,dx, \, \forall \, E \subset \Omega, \, \mu(E) < \infty$.

(ii) *if $p = 1$ and $\mu(\Omega) < \infty$, then the equivalence in (i) also holds.*

(iii) *if $p = 1$ and $\mu(\Omega) = \infty$, then (A)⇒(B) but (B)⇏(A).*

Proof. We first prove (i).

(A)⇒(B) The boundness of $||f_n||_p$ can be inferred by Banach-Steinhaus theorem. Then in the definition of weak convergence in L^p, taking $g(x) = \boldsymbol{\chi}_E(x)$.

(B)⇒(A) We aim to prove the following limit

$$\lim_{n\to\infty} \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx, \, \forall \, g \in L^{p'}$$

hold. Now we present two methods.

Approach 1 In the beginning, we assume that g is a step function of compact support, that is, $g(x) = \sum_{i=1}^{k} a_i \boldsymbol{\chi}_{E_i}(x)$, where $\mu(E_i) < \infty$, then

$$\begin{aligned}\int_\Omega f_n(x)g(x)dx &= \sum_{i=1}^{k} a_i \int_\Omega f_n(x)\boldsymbol{\chi}_{E_i}(x)dx \\ &= \sum_{i=1}^{k} a_i \int_{E_i} f_n(x)dx \to \sum_{i=1}^{k} a_i \int_{E_i} f(x)dx = \int_\Omega f(x)g(x)dx, \text{ as } n \to \infty.\end{aligned}$$

If $g \in L^{p'}$, by density, there exist a sequence of step funcions (g_m) of compact support such that $g_m \to g$ strongly in $L^{p'}$. Then from

$$\left|\int_\Omega f_n g - fg\,dx\right| \le \left|\int_\Omega f_n g - f_n g_m\,dx\right| + \left|\int_\Omega f_n g_m - f g_m\,dx\right| + \left|\int_\Omega f g_m - fg\,dx\right|$$
$$\equiv I_1 + I_2 + I_3,$$

the boundness of $\{||f_n||_p\}_{n\ge1}$ and Hölder inequality we know that given $\varepsilon > 0$, $\exists\, M \in \mathbb{N}$ such that $I_1,\, I_3 \le \dfrac{\varepsilon}{3}$ whenever $m \ge M$. Fix $m \ge M$, $\exists\, N \in \mathbb{N}$, if $n > N$, then $I_2 \le \dfrac{\varepsilon}{3}$, which completes the proof of (i).

Approach 2 Now we devote to prove (ii), similarly, it is easy to prove (A)⇒(B). Arguing by contradiction, if there exist a subsequence (f_{n_k}), $\varepsilon_0 > 0$ and $g_0 \in L^{p'}$ such that

$$\int_\Omega f_{n_k} g_0\,dx \ge \int_\Omega f g_0\,dx + \varepsilon_0,\ \forall\, k \in \mathbb{N}_+, \tag{2.2}$$

meanwhile, since $||f_{n_k}||_p \le M$, then there exist a subsequence $(f_{n_{k_j}})$ and a function $h \in L^p$ such that $f_{n_{k_j}} \rightharpoonup h$ weakly $\sigma(L^p, L^{p'})$. From (B) and the definition of weak convergence we infer that $\forall\, E \subset \Omega$, $\mu(E) < \infty$,

$$\int_E f\,dx = \int_E h\,dx,$$

which implies that $f = h$, $a.e.$ on Ω, then $f_{n_{k_j}} \rightharpoonup f$ weakly $\sigma(L^p, L^{p'})$, which contradicts to (2.2).

For the proof of (ii), we use the fact that the set of simple functions are dense in L^∞ and a similar proof as (i). The counterexample of (iii) we refer to Remark 2.3. □

Remark 2.13 *The conclusion of Theorem 2.6 may fail if $r = q$, that is, (2.1) cannot infer $||f_n - f||_q \to 0$, for instance, let $p = 1$ and $r = q = \infty$, then see the examples in Remark 2.5 (1).*

Theorem 2.14 *([1, Theorem 4.9]) Let (f_n) be a subsequence in L^p, $1 \le p \le \infty$ and let $f \in L^p$ be such that $||f_n - f||_p \to 0$. Then there exist a subsequence (f_{n_k}) and a function $h \in L^p$ such that*

(i) $f_{n_k}(x) \to f(x)$, $a.e. x \in \Omega$.

(ii) $|f_{n_k}(x)| \le h(x)$, $\forall\, k \in \mathbb{N}$, $a.e. x \in \Omega$.

Remark 2.15 *There are other relationships lie in convergence methods, we only list some of them and omit the proofs here.*

(1) *Assume that (f_n) converges in measure to f, meanwhile,*

$$|f_n(x) - f_n(y)| \le M|x - y|,\ x, y \in \Omega.$$

Then there exists a measurable function f such that $f_n \to f$ $a.e. x \in \Omega$.

(2) *Let* (f_n) *be a nonnegative sequence in* L^p, $1 < p < \infty$, *then* $f_n \to f$ *strongly in* L^p *if and only if* $f_n^p \to f^p$ *strongly in* L^1.

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[5] 实分析讲义